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Exponential growth of Betti numbers[1]

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Abstract

We prove over some local commutative noetherian rings that the sequence of Betti numbers of every finitely generated module is either eventually constant or has termwise exponential growth. © 1998 Elsevier Science B.V.

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1. Introduction

Throughout the paper $(R, \mathfrak{m}, k)$ will be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let M be a finitely generated R-module and let

$$\boldsymbol{F}: \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

be its minimal free resolution over R. Then $b_n^R(M) = \operatorname{rank} F_n = \dim_k \operatorname{Tor}_n(M, k)$ is called the nth Betti number of M over R. Avramov addressed the following basic questions:

Question A. What are the possible types of asymptotic behavior of the Betti numbers?

Question B. Is the sequence of Betti numbers of each finitely generated R-module eventually nondecreasing?

A survey by Avramov of the results concerning these questions can be found in [3]. In order to describe the asymptotic growth of the Betti numbers, we use the notion

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of termwise exponential growth, introduced by Fan in [8]. A sequence $\{b_n\}_{n\geq 0}$ of positive real numbers has termwise exponential growth of rate $1 < A \leq \infty$, if $A = \sup_B\{B \in \mathbb{R} \mid \text{there is an integer } n_B \in \mathbb{N}, \text{ such that } Bb_{n-1} \leq b_n \text{ for } n \geq n_B\}$. Note that if a sequence $\{b_n\}_{n\geq 0}$ of Betti numbers of a module has termwise exponential growth, then it has strong exponential growth, i.e. there exist real numbers $1 < C \leq D$ such that $C^n \leq b_n \leq D^n$ for all sufficiently large n. (For the existence of the upper bound cf. e.g. [1].)

The behavior of Betti numbers of modules over a complete intersection is described at length in [4, Sections 7 and 8]. In this paper, we consider the cases when R is not a complete intersection and satisfies one of the conditions in the next Theorem 1. These rings have already been studied in [1, 5, 6, 10–12] and positive answers to Question B are known except in case (b) and when R is Cohen–Macaulay of $\operatorname{mult}(R) = 8$. We give more precise information on the asymptotic growth of the Betti numbers. Our main result is the following:

Theorem 1. *Let R satisfy one of the conditions:*

(a) *R is a Cohen–Macaulay ring of* $\operatorname{mult}(R) \leq 8$,

(b) *R is a Gorenstein ring of* $\operatorname{mult}(R) \leq 11$,

(c) *R is a Golod ring,*

(d) $\mathfrak{m}^3 = 0$ *and* $\{b_n^R(k)\}_{n\geq 0}$ *has termwise exponential growth.*

Let M be a finitely generated R-module. If R is not a complete intersection then the sequence $\{b_n^R(M)\}_{n\geq 0}$ either is eventually constant or has termwise exponential growth.

The proof of the theorem is divided into three parts because in the different cases we use different techniques and notation: cases (a) and (b) are considered in Section 2, case (c) is in Section 3, and case (d) is in Section 4.

The Betti numbers grow either polynomially or exponentially in all cases when their behavior is understood. For example, they grow polynomially over a complete intersection. In contrast, by Theorem 1 and [12, (6.5)] it follows that the Betti numbers grow termwise exponentially over a Golod ring which is not a hypersurface.

2. Rings of small multiplicity

Let R be an artinian ring and M be a finitely generated R-module. Denote by $\nu(M)$ the minimal number of generators of M, and set $e = \operatorname{edim} R = \nu(\mathfrak{m})$, $l = \operatorname{length} R$, $h = \min\{i \mid \mathfrak{m}^{i+1} = 0\}$, and $b_n = b_n^R(M)$. In [10, (2.2) and (2.3)] we proved:

Proposition 2. *Let R be an artinian ring and M a finitely generated R-module.*

(i) *If $2e-l+h-1 = 1$, then there exists an integer p satisfying $\nu(M)-1 \leq p \leq \infty$, such that $b_n = b_{n+1}$ for* $\max\{\nu(M)-1, 1\} \leq n < p$ *and $b_n < b_{n+1}$ for $n \geq p$.*

(ii) *If $2e - l + h - 1 \geq 2$, then $\{b_n\}_{n\geq 0}$ has termwise exponential growth.*

In order to study Betti numbers over rings of small multiplicity, we first obtain further information in the case of Proposition 2(i):

Proposition 3. *Under the conditions of Proposition* 2(i), *suppose that* $e \geq 3$ *and* $p < \infty$. *Then there is a real number* $C > 1$, *such that* $b_{n+1} \geq Cb_n$ *for* $n \geq p$.

Proof. Set $C = \min\{b_{p+1}/b_p, e-1\} > 1$. We will prove by induction on n that $b_{n+1} \geq Cb_n$. By the choice of C, the inequality $b_{p+1} \geq Cb_p$ holds. Assume that $b_n \geq Cb_{n-1}$ for some $n \geq p+1$. The following inequality $b_{n+1} \geq eb_n - (\text{length}\,(\mathfrak{m}^2) + 2 - h)b_{n-1}$ was established in [10, (2.2), inequality $(*)$]. By assumption $h = l - 2e + 2$, so we obtain that

$$\begin{aligned} b_{n+1} &\geq eb_n - (\text{length}\,(\mathfrak{m}^2) + 2 - h)C^{-1}b_n \\ &= eb_n + [(l-1-e) + 2 - (l-2e+2)]C^{-1}b_n \\ &= eb_n + (1-e)C^{-1}b_n \\ &= (Ce + 1 - e)C^{-1}b_n \\ &= [C - (C^2 - Ce + (e-1))C^{-1}]b_n \\ &= [C - (C-1)(C-(e-1))C^{-1}]b_n \\ &\geq Cb_n, \end{aligned}$$

where the last inequality is implied by the inequalities $C > 1$ and $C \leq e - 1$. □

Remark 4. Let R be an artinian ring with $\operatorname{edim} R \leq 3$ or an artinian Gorenstein ring with $\operatorname{edim} R = 4$. Let M be a finitely generated R-module. If R is not a complete intersection then $\{b_n^R(M)\}_{n\geq 0}$ either is eventually constant or has termwise exponential growth.

Proof. Let ρ_M be the radius of convergence of the Poincaré series of M. When $\rho_M < 1$ by [14, (1.2)] the Betti sequence $\{b_n^R(M)\}_{n\geq 0}$ has termwise exponential growth. Suppose $\rho_M \geq 1$. We can assume that k is infinite and R is complete (if necessary replace R by the maximal-ideal-adic completion of $R[Y]_{\mathfrak{m}R[Y]}$, where Y is an indeterminate over R). Since R is not a complete intersection, it follows from [1, Proof of Theorem (3.1) in cases (a) and (b)] that $R \cong Q/(x)$, where Q is a local ring such that $\operatorname{pd}_Q M < \infty$ and x is a nonzero divisor. By [2, (4.1)] the sequence $\{b_n^R(M)\}_{n\geq 0}$ is eventually constant. □

Now we are ready for the proof of Theorem 1.

Proof of Theorem 1 (Cases (a) and (b)). Replacing (if necessary) R by $R' = R[X]_{\mathfrak{m}R[X]}$ (where X is an indeterminate), and M by $M \otimes_R R'$ we may assume that k is infinite. Set $d = \dim R$. Then there exists an R-regular sequence $a_1, \ldots, a_d$, such that the multiplicity of R is equal to the length of the artinian ring $\bar{R} = R/(a_1, \ldots, a_d)$, and

$b^R_{n+d}(M) = b^{\bar{R}}_n(\mathrm{Syz}^R_d(M)\otimes_R \bar{R})$ for $n \geq 0$. So it is sufficient to consider artinian rings of length ≤ 8 and artinian Gorenstein rings of length ≤ 11.

If $h = 1$ apply Lemma 6, so we can assume that $h \geq 2$.

Case (a): The desired result follows by Propositions 2 and 3 if $2e - l + h - 1 \geq 1$. This condition is satisfied when $e \geq 4$ because $2e - l + h - 1 \geq 8 - 8 + 2 - 1 = 1$. Thus, it remains to consider the case when $e \leq 3$. Then Remark 4 applies.

Case (b): Again the result follows by Propositions 2 and 3 if $2e - l + h - 1 \geq 1$. This inequality holds when $h \geq 3$ and $e \geq 5$, since then $2e - l + h - 1 \geq 10 - 11 + 3 - 1 = 1$. So it remains to consider the cases when either $e \leq 4$ or $h = 2$. In the former case Remark 4 can be applied. The latter case is settled by Sjödin in [13, Proofs of Lemma 5 and Theorem].

3. Golod rings

Proof of Theorem 1 (Case (c)). When R is a hypersurface ring by a result of Eisenbud [7, (6.1)] the Betti sequence of any module is eventually constant. Otherwise, as was proved by Lescot [12, (6.5)] the sequence $\{b^R_n(M)\}_{n\geq 0}$ is either eventually zero or eventually strictly increasing. We give the following more presice information.

Proposition 5. *Let R be a Golod ring, which is not a hypersurface, and M a finitely generated R-module. Set*

$$B = \min\left(\frac{b^R_{2\operatorname{edim} R}(M)}{b^R_{2\operatorname{edim} R-1}(M)}, \ldots, \frac{b^R_{3\operatorname{edim} R-1}(M)}{b^R_{3\operatorname{edim} R-2}(M)}\right).$$

Then $B > 1$, *and* $b^R_{n+1}(M) \geq B b^R_n(M)$ *for* $n \geq 2\operatorname{edim} R$.

Proof. Let S be a regular local ring, and I be an ideal in the square of the maximal ideal of S, such that $R = S/I$; set $e = \operatorname{edim} R$. We will consider the eth syzygy module $N = \mathrm{Syz}^R_e(M)$. Set $b_n = b^R_n(N)$, then we have equalities of Betti numbers $b_n = b^R_{n+e}(M)$ so it is enough to bound the Betti numbers of N. The advantage of passing to N is that by [12, (6.3)(b)] the Poincaré series of N, $P^N_R(t) = \sum_{n=0}^{\infty} b_n t^n$, can be written in the form

$$P^N_R(t) = \frac{P^N_S(t)}{1 + t - tP^R_S(t)}.$$

The vanishing of the Euler characteristic for the minimal free resolutions of N and R over S implies that both $f(t) = P^N_S(t)/(1+t)$ and $g(t) = (1 + t - tP^R_S(t))/1 + t$ are polynomials. Set $c_n = b^S_n(R)$, then $g(t)$ can be written in the form $g(t) = 1 - t - a_2t^2 - \cdots - a_rt^r$, where $a_n = \sum_{i\geq 1}^{n}(-1)^{i+1}c_{n-i}$ are integer numbers, $a_r \neq 0$, and

$2 \leq r \leq e$. Denote by K the field of s of S, then $a_n = \dim_K((\mathrm{Syz}_n^S(R))_{(0)})$ are nonnegative numbers. Noting that

$$P_R^N(t) = \frac{f(t)}{1 - t - a_2t^2 - \cdots - a_rt^r}$$

and $\deg k(f(t)) < e$, we see that the Betti numbers of N satisfy the relation

$$b_n = b_{n-1} + a_2b_{n-1} + \cdots + a_rb_{n-r} \quad \text{for } n \geq e.$$

This shows that the Betti numbers are strictly increasing after e steps, and $B > 1$.

Next, we will prove the termwise exponential growth by induction. By the definition of B we have that $b_i \geq Bb_{i-1}$ for $e \leq i \leq 2e - 1$. Suppose that for some $n \geq 2e$ we have already proved the inequalities $b_i \geq Bb_{i-1}$ for $e \leq i \leq n - 1$. Then,

$$\begin{aligned} b_n &= b_{n-1} + a_2b_{n-2} + \cdots + a_rb_{n-r} \\ &\geq Bb_{n-2} + a_2Bb_{n-3} + \cdots + a_rBb_{n-r-1} = Bb_{n-1}. \quad \square \end{aligned}$$

Remark. $P_R^M(t)$ can be written with denominator $1+t-tP_S^R(t) = 1-c_1t^2-\cdots-c_rt^{r+1}$. The equation $1 = c_1t^2 + \cdots + c_rt^{r+1}$ has one positive real root, say α, and $0 < \alpha < 1$. The inequality $|c_1\beta^2 + \cdots + c_r\beta^{r+1}| \leq c_1|\beta^2| + \cdots + c_r|\beta^{r+1}|$ implies that there is no complex root $\beta \neq \alpha$ with $|\beta| = \alpha$. So by a lemma of Sun [14, Lemma 2], the sequence $\{b_n/b_{n+1}\}_{n\geq 0}$ converges, which implies "eventual" termwise exponential growth of the Betti numbers. This remark was observed independently by Sun.

4. Rings with $\mathfrak{m}^3 = 0$

We want to remark that by [11, Theorem B and (3.9)] there might exist a nonzero module with constant Betti numbers only if R is Koszul and $\dim_k \mathfrak{m}^2 = \operatorname{edim} R - 1$.

Proof of Theorem 1 (Case (d)). First, we consider the case when $\operatorname{socle}(\mathfrak{m}) = \mathfrak{m}^2$. According to [11, (3.1)], an R-module M is called exceptional if k is not a direct summand of any syzygy of M. A proof of the fact that the theorem holds for exceptional modules can be extracted from the proof of [11, (3.7), Case (b)]. So it suffices to consider nonexceptional modules.

If k is a direct summand of $\mathrm{Syn}_n^R(M)$, replacing M by $\mathrm{Syz}_n^R(M)$ we may assume that $M = M' \oplus k$. By assumption, there is a real number $A > 1$ and an integer $n_1 > 0$, such that the inequality $b_n^R(k) \geq Ab_{n-1}^R(k)$ holds for $n \geq n_1$. From [11, Theorem B] there is an integer $n_2 > 0$, such that the sequence $\{b_n^R(M')\}_{n\geq n_2}$ is nondecreasing. Set $n_0 = \max(n_1, n_2 + 1)$. On the other hand, by [11, Lemma in the preliminaries] there is a real number $D \geq 1$, such that $b_n^R(M') \leq Db_n^R(k)$ for $n \geq 0$. Set $C = (A + D)/(1 + D) > 1$.

Then for $n \geq n_0$ the following (in)equalities hold:

$$\begin{aligned} b_n^R(M) &= b_n^R(k) + b_n^R(M') \\ &\geq Ab_{n-1}^R(k) + b_{n-1}^R(M') \\ &= Cb_{n-1}^R(k) + Cb_{n-1}^R(M') + (A-C)b_{n-1}^R(k) - (C-1)b_{n-1}^R(M') \\ &= Cb_{n-1}^R(M) + \frac{A-1}{1+D}Db_{n-1}^R(k) - \frac{A-1}{1+D}b_{n-1}^R(M') \\ &= Cb_{n-1}^R(M) + \frac{A-1}{1+D}(Db_{n-1}^R(k) - b_{n-1}^R(M')) \\ &\geq Cb_{n-1}^R(M), \end{aligned}$$

where the last inequality holds because $A > 1$ and $b_{n-1}^R(M') \leq Db_{n-1}^R(k)$.

It remains to consider the case when $\operatorname{socle}(\mathfrak{m}) \neq \mathfrak{m}^2$. Then we apply the following:

Lemma 6. *Let $(0:\mathfrak{m}) \not\subseteq \mathfrak{m}^2$ and $\operatorname{edim} R \geq 2$. If M is a nonfree finitely generated R-module then $\{b_n^R(M)\}_{n\geq 1}$ is strictly increasing and has strong exponential growth. If in addition the ring R is artinian, then $\{b_n^R(M)\}_{n\geq 0}$ has termwise exponential growth of rate $\geq 1 + 1/(\operatorname{length}(R) - 1)$.*

Proof. Choose $f \in (0:\mathfrak{m})$, $f \notin \mathfrak{m}^2$. It follows from $\operatorname{edim} R \geq 2$ that there exists a $g \in \mathfrak{m}$, $g \notin \mathfrak{m}^2$, for which the images of f and g are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$.

Let $\boldsymbol{F}$ be a minimal free resolution of M over R. Set $f_i^{(n)} = (0,\dots,0,f,0,\dots,0) \in F_n \cong R^{b_n}$ (f is in the ith position). Note that the $f_i^{(n)}$ (for $1 \leq i \leq b_n$) are in $\operatorname{Ker} d_n$. Besides, their images are linearly independent in $(\operatorname{Ker} d_n/\mathfrak{m}\operatorname{Ker} d_n)$. So the $f_i^{(n)}$ are contained in a minimal generating set for $\operatorname{Ker} d_n$. Constructing the basis of $F_{n+1} \cong R^{b_{n+1}}$ accordingly, we obtain the map d_{n+1} as given by a matrix in the form

$$\begin{pmatrix} f & 0 & \cdots & 0 & \cdots \\ 0 & f & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & f & \cdots \end{pmatrix}.$$

This shows that $g_i^{(n+1)} = (0,\dots,0,g,0,\dots,0) \in F_{n+1}$ (g is in the ith position), $1 \leq i \leq b_n$, are in $\operatorname{Ker} d_{n+1}$. Note that the images of $f_i^{(n+1)}$, $1 \leq i \leq b_{n+1}$, and $g_j^{(n+1)}$, $1 \leq j \leq b_n$, are linearly independent in $\operatorname{Ker} d_{n+1}/\mathfrak{m}\operatorname{Ker} d_{n+1}$. Thus, $f_i^{(n+1)}$ and $g_j^{(n+1)}$ are contained in a minimal generating set for $\operatorname{Ker} d_{n+1}$. It follows that $b_{n+2} \geq b_{n+1} + b_n$ for every $n \geq 1$. Therefore, b_n is not less than the $(n-1)$th Fibonacci number $(\sqrt{5}/10.2^n)((1+\sqrt{5})^n - (1-\sqrt{5})^n)$. So $b_n \geq (\frac{5}{4})^n$ for $n \geq 3$.

As $b_{n+1} = \nu(\operatorname{Syz}_{n+1}^R(M))$ and $\operatorname{Syz}_{n+1}^R(M) \subset \mathfrak{m}R^{b_n}$ we have the inequality $b_{n+1} \leq b_n(\operatorname{length}(R) - 1)$. Hence, we get

$$b_{n+2} \geq b_{n+1} + b_n \geq \left(1 + \frac{1}{\operatorname{length}(R) - 1}\right) b_{n+1}. \qquad \square$$

Example. Let $R = k[x,y,z,v]/I$, where $I = (x^2, y^2, z^2, v^2, xy, zv)$. Then, R is an artinian ring with $\mathfrak{m}^3 = 0$. As I is generated by quadratic monomials, R is Koszul. Hence, by [9, Section 5, Corollary 1]

$$P_R(t) = \frac{1}{\mathrm{Hilb}_R(-t)} = \frac{1}{1-4t+4t^2} = \frac{1}{(1-2t)^2} = \sum_{n=0}^{\infty} \binom{n+1}{n} 2^n t^n$$
$$= \sum_{n=0}^{\infty} (n+1) 2^n t^n.$$

So $b_{n+1}^R(k) = 2^{n+1}(n+2) > 2^{n+1}(n+1) = 2b_n^R(k)$. Thus, the sequence $\{b_n^R(k)\}_{n \ge 0}$ has termwise exponential growth. By [11, Theorem B] there exist no nonzero R-modules with constant Betti numbers. By Theorem 1 $\{b_n^R(M)\}_{n\ge 0}$ has termwise exponential growth for any nonfree finitely generated R-module M.

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